Problem Set 8 – Relativity (G0I36A)

1 Free Scalar Fields in Curved Spacetime

1.) Remember, when deriving the Euler-Lagrange equations we have to treat ϕ and its derivatives as independent quantities. The differentiation with respect to ϕ is easy:

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \ . \tag{1.1}$$

The derivative with respect to $\nabla_\mu \phi$ is given by

$$\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \phi)} = -\nabla^\mu \phi \ . \tag{1.2}$$

Putting this all together, we find the Euler-Lagrange equation

$$-m^2\phi + \nabla_\mu(\nabla^\mu \phi) = 0 \ , \tag{1.3}$$

which is precisely the Klein-Gordon equation.

2.) Using the chain rule, we first note that

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} = \frac{1}{\sqrt{-g}}\left(-\frac{1}{2\sqrt{-g}}\partial_\mu g\right) = \frac{1}{2g}\partial_\mu g \ . \tag{1.4}$$

Derivatives are linear operators, and $g_{\mu\nu}$ can be thought of as a matrix with inverse $g^{\mu\nu}$, and thus we can apply Jacobi's formula:

$$\partial_\mu g = g \, \mathrm{tr}(g^{\nu\rho}\partial_\mu g_{\rho\sigma}) \ . \tag{1.5}$$

Note that we had to contract the inner indices on $g^{\nu\rho}$ and $g_{\rho\sigma}$; this is how we multiply them together as matrices. Then, taking the trace simply corresponds to contracting over the outer indices, and so we are left with

$$\partial_\mu g = g \, g^{\nu\rho}\partial_\mu g_{\rho\nu} \ . \tag{1.6}$$

Putting this all together, we find that

$$\frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} = \frac{1}{2} g^{\nu\rho}\partial_\mu g_{\rho\nu} \ . \tag{1.7}$$

The Christoffel symbols are defined by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \ , \tag{1.8}$$

and so contracting over the ρ and ν indices yields

$$\Gamma^\nu_{\mu\nu} = \frac{1}{2} g^{\nu\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \ . \tag{1.9}$$

Notice that $g^{\nu\sigma}$ is symmetric under $\nu \leftrightarrow \sigma$, but $\partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}$ is antisymmetric under the same exchange. Contracting a symmetric tensor with an antisymmetric tensor yields zero, as we proved on problem set 1, and so the second and third term drop out. We are left with

$$\Gamma^{\nu}_{\mu\nu} = \frac{1}{2} g^{\nu\sigma} \partial_{\mu} g_{\nu\sigma} \ . \tag{1.10}$$

If we compare (1.7) and (1.10), we can see (after a relabeling of dummy indices) that the expressions are equal, and thus

$$\boxed{\frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} = \Gamma^{\nu}_{\mu\nu}} \ . \tag{1.11}$$

Now, we note that $\nabla^{\mu}\phi$ transforms as a $(1,0)$ tensor, and so a covariant derivative acting on it yields

$$\begin{aligned} \Box\phi &= \nabla_{\mu}(\nabla^{\mu}\phi) \\ &= \partial_{\mu}(\nabla^{\mu}\phi) + \Gamma^{\mu}_{\mu\nu}(\nabla^{\nu}\phi) \\ &= \partial_{\mu}(\nabla^{\mu}\phi) + \frac{1}{\sqrt{-g}}(\partial_{\nu}\sqrt{-g})\nabla^{\nu}\phi \\ &= \frac{\sqrt{-g}}{\sqrt{-g}}\partial_{\mu}(\nabla^{\mu}\phi) + \frac{1}{\sqrt{-g}}(\partial_{\mu}\sqrt{-g})\nabla^{\mu}\phi \\ &= \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\,\nabla^{\mu}\phi) \\ &= \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\,g^{\mu\nu}\partial_{\nu}\phi) \ , \end{aligned} \tag{1.12}$$

where in the third line we used the intermediate result (1.11), and then we played around to combine terms under a common derivative, and then in the last line we used the fact that $\nabla_{\nu}\phi = \partial_{\nu}\phi$. This completes the proof.

3.) From the previous problem, we know that the Klein-Gordon operator $\Box$ acting on the field ϕ can be written as

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\,g^{\mu\nu}\partial_{\nu}\phi) \ . \tag{1.13}$$

We want to now consider the Minkowski metric in spherical coordinates:

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\psi^2 \ , \tag{1.14}$$

where we are using ψ as our azimuthal coordinate, since ϕ is already being used for the scalar field. For this metric, we can easily compute the determinant

$$\sqrt{-g} = cr^2 \sin\theta \ . \tag{1.15}$$

Importantly, the metric is diagonal, and so the sum over the indices μ and ν inside (1.13) will yield zero unless $\mu = \nu$. We therefore have four relevant terms in the sum:

$$\begin{aligned} \Box\phi = \ & \frac{1}{cr^2\sin\theta}\partial_t\left(cr^2\sin\theta\,g^{tt}\partial_t\phi\right) + \frac{1}{cr^2\sin\theta}\partial_r\left(cr^2\sin\theta\,g^{rr}\partial_r\phi\right) \\ & + \frac{1}{cr^2\sin\theta}\partial_\theta\left(cr^2\sin\theta\,g^{\theta\theta}\partial_\theta\phi\right) + \frac{1}{cr^2\sin\theta}\partial_\psi\left(cr^2\sin\theta\,g^{\psi\psi}\partial_\psi\phi\right) \ . \end{aligned} \tag{1.16}$$

Note that the speed of light c is constant and so we can pull it outside of all derivatives, at which point it cancels. Additionally, the diagonal nature of the metric means the inverse metric components are simple to compute:

$$\begin{aligned} g^{tt} &= -\frac{1}{c^2} \ , \\ g^{rr} &= 1 \ , \\ g^{\theta\theta} &= \frac{1}{r^2} \ , \\ g^{\psi\psi} &= \frac{1}{r^2\sin^2\theta} \ . \end{aligned} \tag{1.17}$$

Plugging these into (1.16), we find that

$$\begin{aligned}\Box\phi &= \frac{1}{r^2\sin\theta}\partial_t\left(-\frac{r^2}{c^2}\sin\theta\,\partial_t\phi\right) + \frac{1}{r^2\sin\theta}\partial_r\left(r^2\sin\theta\,\partial_r\phi\right) \\ &\quad + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\phi\right) + \frac{1}{r^2\sin\theta}\partial_\psi\left(\frac{1}{\sin\theta}\partial_\psi\phi\right)\ .\end{aligned} \tag{1.18}$$

Finally, all we have to do is note that the coordinates are independent, and so we can pull r and θ outside of the ∂_t derivative, θ out of the ∂_r derivative, etc. Thus, the Klein-Gordon operator becomes much simpler:

$$\Box\phi = -\frac{1}{c^2}\partial_t^2\phi + \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\phi\right) + \frac{1}{r^2\sin^2\theta}\partial_\psi^2\phi\ . \tag{1.19}$$

Therefore, the Klein-Gordon equation $(\Box - m^2)\phi = 0$ can be written in spherical coordinates as

$$-\frac{1}{c^2}\partial_t^2\phi + \frac{1}{r^2}\partial_r\left(r^2\partial_r\phi\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\phi\right) + \frac{1}{r^2\sin^2\theta}\partial_\psi^2\phi - m^2\phi = 0\ . \tag{1.20}$$

4.) The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2dt^2 + \frac{dr^2}{1-\frac{2GM}{c^2r}} + r^2d\Omega^2\ , \tag{1.21}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\,d\psi^2$ is our metric on S^2. As the problem recommends, we will set $c = 1$ from here on out for simplicity. To write down the Klein-Gordon equation explicitly on the Schwarzschild background, we first note that

$$\sqrt{-g} = r^2\sin\theta\ , \tag{1.22}$$

and that the inverse metric components are

$$g^{tt} = 1/g^{rr} = \left(1 - \frac{2GM}{r}\right)^{-1}\ ,\quad g^{\theta\theta} = \frac{1}{r^2}\ ,\quad g^{\psi\psi} = \frac{1}{r^2\sin^2\theta}\ . \tag{1.23}$$

Armed with this, it is straightforward but tedious to compute that

$$\Box\phi = -\frac{r}{r-2GM}\frac{d^2\phi}{dt^2} + \frac{1}{r^2}\frac{d}{dr}\left((r^2 - 2GMr)\frac{d\phi}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\phi}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\phi}{d\psi^2}\ . \tag{1.24}$$

Note that in deriving this, we used the fact that the four coordinates (t, r, θ, ψ) are all independent, and so e.g. $\frac{dr}{dt} = 0$, etc. The full Klein-Gordon equation is thus given by

$$0 = \left[-\frac{r}{r-2GM}\frac{d^2}{dt^2} + \frac{1}{r^2}\frac{d}{dr}\left((r^2 - 2GMr)\frac{d}{dr}\right) + \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2}{d\psi^2} - m^2\right]\phi\ , \tag{1.25}$$

where $\phi = \phi(t, r, \theta, \psi)$ is some as-of-yet undetermined function of all spacetime coordinates.

2 The Einstein Equation with a Free Scalar

5.) First, note that the variation δ is a linear operator, and so Jacobi's formula applies when computing δg. Using the chain rule and Jacobi's formula, we find that:

$$\delta\sqrt{-g} = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}g\,g^{\mu\nu}\delta g_{\mu\nu} = \frac{\sqrt{-g}}{2}g^{\mu\nu}\delta g_{\mu\nu}\ . \tag{2.1}$$

In order to find the variation with respect to $\delta g^{\mu\nu}$, we need to figure out how to express $\delta g_{\mu\nu}$ in terms of $\delta g^{\mu\nu}$. To do so, we recall that, since $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, it follows that

$$g_{\mu\rho}g^{\rho\nu} = \delta_\mu^\nu \ , \tag{2.2}$$

i.e. their contraction gives the Kronecker delta symbol. This symbol is constant, and so its variation is zero. Thus we find that

$$0 = \delta(g_{\mu\rho}g^{\rho\nu}) = g_{\mu\rho}\delta g^{\rho\nu} + g^{\rho\nu}\delta g_{\mu\rho} \ . \tag{2.3}$$

Contracting this expression with $g_{\nu\sigma}$ then tells us that $0 = \delta g_{\mu\sigma} + g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\nu}$, which we can rewrite slightly as

$$\delta g_{\mu\nu} = -g_{\mu\rho}g_{\nu\sigma}\delta g^{\rho\sigma} \ . \tag{2.4}$$

Plugging this into (2.1), we find that

$$\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\mu\nu} \ . \tag{2.5}$$

Now, we can use the chain rule to compute the variation of the matter action with respect to the metric:

$$\begin{aligned}
\delta\left(\sqrt{-g}\,\mathcal{L}_M\right) &= \delta\left[\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right)\right] \\
&= (\delta\sqrt{-g})\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right) + \sqrt{-g}\left(-\frac{1}{2}\delta g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) \\
&= \sqrt{-g}\left[-\frac{1}{2}g_{\rho\sigma}\delta g^{\rho\sigma}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right) + \left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right)\delta g^{\mu\nu}\right] \\
&= \sqrt{-g}\left[-\frac{1}{2}g_{\mu\nu}\delta g^{\mu\nu}\left(-\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi - \frac{1}{2}m^2\phi^2\right) + \left(-\frac{1}{2}\nabla_\mu\phi\nabla_\nu\phi\right)\delta g^{\mu\nu}\right] \\
&= \frac{1}{2}\sqrt{-g}\left[\frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + m^2\phi^2\right) - \nabla_\mu\phi\nabla_\nu\phi\right]\delta g^{\mu\nu} \ .
\end{aligned} \tag{2.6}$$

Since we have explicitly put $\delta g^{\mu\nu}$ on the far right-hand side of this expression, it is easy to compute the stress-energy tensor:

$$\begin{aligned}
T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\,\mathcal{L}_M)}{\delta g^{\mu\nu}} \\
&= -\frac{2}{\sqrt{-g}}\left(\frac{1}{2}\sqrt{-g}\left[\frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + m^2\phi^2\right) - \nabla_\mu\phi\nabla_\nu\phi\right]\right) \\
&= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + m^2\phi^2\right) \ .
\end{aligned} \tag{2.7}$$

That is, this is the stress-energy tensor for a free scalar of mass m.

Now, we need to prove that this stress tensor is conserved. If we use the chain rule and remember that $\nabla_\mu g_{\rho\sigma} = 0$, we find that

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= \nabla_\mu\left(\nabla^\mu\phi\nabla^\nu\phi - \frac{1}{2}g^{\mu\nu}(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + m^2\phi^2)\right) \\
&= \nabla_\mu(\nabla^\mu\phi\nabla^\nu\phi) - \frac{1}{2}\nabla^\nu(g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi + m^2\phi^2) \\
&= (\Box\phi)(\nabla^\nu\phi) + (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - \frac{1}{2}g^{\rho\sigma}(\nabla^\nu\nabla_\rho\phi)(\nabla_\sigma\phi) \\
&\quad - \frac{1}{2}g^{\rho\sigma}(\nabla_\rho\phi)(\nabla^\nu\nabla_\sigma\phi) - m^2\phi\nabla^\nu\phi \\
&= (\Box\phi)(\nabla^\nu\phi) + (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - g^{\rho\sigma}(\nabla^\nu\nabla_\rho\phi)(\nabla_\sigma\phi) - m^2\phi\nabla^\nu\phi \ ,
\end{aligned} \tag{2.8}$$

where in the last line we used the symmetry of the metric tensor to combine like terms. If we use the equation of motion for ϕ, we know that $\Box\phi = m^2\phi$, and so the first and last terms in the above expression cancel. We are left with:

$$\begin{aligned}
\nabla_\mu T^{\mu\nu} &= (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - g^{\rho\sigma}(\nabla^\nu\nabla_\rho\phi)(\nabla_\sigma\phi) \\
&= (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - (\nabla^\nu\nabla_\rho\phi)(g^{\rho\sigma}\nabla_\sigma\phi) \\
&= (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - (\nabla^\nu\nabla_\rho\phi)(\nabla^\rho\phi) \\
&= (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - (\nabla^\rho\phi)(\nabla^\nu\nabla_\rho\phi) \\
&= (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi) - (\nabla^\mu\phi)(\nabla^\nu\nabla_\mu\phi) \\
&= (\nabla^\mu\phi)(\nabla_\mu\nabla^\nu\phi - \nabla^\nu\nabla_\mu\phi) \\
&= (\nabla_\mu\phi)(\nabla^\mu\nabla^\nu\phi - \nabla^\nu\nabla^\mu\phi) \ .
\end{aligned} \tag{2.9}$$

Importantly, covariant derivatives commute when acting on a scalar field. This can be seen explicitly by expanding in terms of Christoffel symbols:

$$\nabla_\mu\nabla_\nu\phi = \partial_\mu\partial_\nu\phi - \Gamma^\rho_{\mu\nu}\partial_\rho\phi \ , \tag{2.10}$$

and then recalling that the Christoffel symbols are symmetric in their lower index and that partial derivatives commute. Thus, we conclude that

$$\nabla_\mu\nabla_\nu\phi = \nabla_\nu\nabla_\mu\phi \ , \tag{2.11}$$

and thus second term at the end of (2.9) vanishes. We are thus left with $\nabla_\mu T^{\mu\nu} = 0$, so the stress-energy tensor for our scalar field is conserved.

6.) If we set $\phi = 0$ everywhere in our spacetime, then it follows that $\nabla_\mu\phi = 0$ as well. Thus, the stress-energy tensor (2.7) becomes identically zero. The Einstein equation thus simplifies to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \ , \tag{2.12}$$

which is identical to the vacuum Einstein equation (i.e. the Einstein equation when no matter is present). We know that the Schwarzschild solution is a solution to the vacuum Einstein equation, and so it is thus also a solution in this case when we set $\phi = 0$.

7.) Both the Einstein equation and the scalar equation of motion will be modified! Remember, we derived the Einstein equation by demanding that

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \ . \tag{2.13}$$

Since the Ricci scalar R now appears with the scalar field in front of it, ϕ will now explicitly show up in the left-hand side of the Einstein equation when we compute $\delta S'$. We will not do this computation explicitly, but you should be able to see what things change now that ϕ is present.

Similarly, the Euler-Lagrange equation for ϕ now gets modified by the additional term

$$\frac{\partial(\phi^2 R)}{\partial\phi} = 2\phi R \ , \tag{2.14}$$

and so the full Euler-Lagrange equation for ϕ becomes

$$(\Box - m^2 + 2R)\phi = 0 \ . \tag{2.15}$$

This is a much more complicated equation to solve than the usual Klein-Gordon equation, since R is in general a fairly complicated object to compute on arbitrary curved backgrounds.

Therefore, adding this new $\phi^2 R$ term has consequences for both the Einstein equation and the ϕ equation of motion. These kinds of theories where the Einstein equation is modified by new terms in the action are referred to as *modified gravity theories*.